

Identification of Parameters through the Approximate Periodic Solutions of a Parabolic System

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Abstract. This work is concerned with the identification problem for what we call the perturbation term or error term in a parabolic partial differential equation, through its approximate periodic solutions. The observation is made over a subregion of the physical domain. The existence and uniqueness problem of the approximate periodic solutions is studied in the first part of the paper. A solution to the identification problem is given in the second part of the paper. The main ingredients to be used include the classical Galerkin method and the more recently developed Carleman estimates for a parabolic system.

Key words. Identification of parameter, Galerkin method, Carleman inequality, approximate periodic solution, parabolic equation.

AMS subject classification. 35K99, 93A99.

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with C^2 -smooth boundary $\partial\Omega$ and let $\omega \subset \Omega$ be a subdomain. Write $Q = \Omega \times (0, T)$ with $T > 0$ and write $Q^\omega = \omega \times (0, T)$. Consider the following parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) + Lu(x, t) = f(x, t), & \text{in } Q = \Omega \times (0, T), \\ u(x, t) = 0, & \text{on } \Sigma = \partial\Omega \times (0, T), \end{cases} \quad (1.1)$$

where

$$Lu(x, t) = L_0u(x, t) + e(x, t)u(x, t),$$

$$L_0u(x, t) = - \sum_{i,j=1}^n D_j(a^{ij}(x)D_iu(x, t)) + \sum_{i=1}^n b^i(x)D_iu(x, t) - \sum_{i=1}^n D_i(b^i(x)u(x, t)) + c(x)u(x, t).$$

Here and in what follows, we write $D_j = \frac{\partial}{\partial x_j}$. We also use the standard summation convention. Namely, repeated indices imply summation from 1 to n . Throughout of the paper, we make the following regularity assumptions for the coefficients:

- (I): $a^{ij}(x) \in Lip(\overline{\Omega})$, $a^{ij}(x) = a^{ji}(x)$, and $\lambda^*|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \frac{1}{\lambda^*}|\xi|^2$, for $\xi \in \mathbf{R}^n$ with λ^* a certain positive constant;
 (II): $b^i(x) \in Lip(\overline{\Omega})$, $c(x) \in L^\infty(\Omega)$, and $e(x, t) \in L^\infty(0, T; L^q(\Omega))$ with $q > \frac{n+2}{2}$ and $f(x, t) \in L^2(Q)$.

In many applications, one often encounters various problems, such as the inverse problem and the Pontryagin maximum principle, related to the periodic solutions of (1.1). (See [11] [15] [16] [17], etc). Here, we recall that a periodic solution of (1.1) is a solution satisfying the following condition:

$$u(x, 0) = u(x, T) \quad \text{in } \Omega. \quad (1.2)$$

In (1.1), the coefficients of the principal part L_0 of the operator L is t -independent. However, the one for what we call the perturbation term or the error term $e(x, t)$ may well depend on the time variable t . It is known that for (1.1), when the operator L is not positive (for instance, when $e(x, t)$ takes negative values), the periodic solution of (1.1) may not exist for a generic choice of $f(x, t)$. (See Example 3.4 in Section 3). Namely, adding an error term with coefficient $e(x, t)$ to the system may well destroy the periodicity of certain solutions even if L_0 is a positive operator. However, as we will show, the system always possesses solutions with certain approximate periodicity. This makes it a natural problem to consider the inverse problem, the Pontryagin problem, and many others, for (1.1) through a certain family of solutions with approximate periodicity. In this paper, we will make an effort towards this study by introducing the concept of approximate periodic solutions through the principle part L_0 of L . We then study the existence and uniqueness of such solutions and use them to identify the error term through the observation of solutions over ω .

We next introduce the concept of \mathcal{K} -approximate periodic solutions of (1.1), where \mathcal{K} is a non-negative integer.

First, we notice that L_0 is a symmetric operator. Consider the eigenvalue problem of L_0 :

$$\begin{cases} L_0 v(x) = \lambda v(x), \\ v(x)|_{\partial\Omega} = 0. \end{cases} \quad (1.3)$$

It is well-known (see [8]) that (1.3) has a complete set of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ with the associate eigenvectors $\{X_j(x)\}_{j=1}^\infty$ such that $L_0 X_j(x) = \lambda_j X_j(x)$, $-\infty < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots < \infty$, $\lim_{j \rightarrow \infty} \lambda_j = \infty$, $X_j(x) \in H_0^1(\Omega)$. Choose $\{X_j(x)\}_{j=1}^\infty$ so that it serves an orthonormal basis of $L^2(\Omega)$. Therefore, $\forall u(x, t) \in L^2(Q)$, we have $u(x, t) = \sum_{j=1}^\infty u_j(t) X_j(x)$, where $u_j(t) = \int_\Omega u(x, t) X_j(x) dx \in L^2(0, T)$.

Definition 1.1. We call $u(x, t)$ is a \mathcal{K} -approximate periodic solution of (1.1) with respect to its principal part L_0 if

(a): $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ is a weak solution of (1.1);

(b): $u \in \mathcal{S}_\mathcal{K} = \{u \in C([0, T]; L^2(\Omega)); u_j(0) = u_j(T) \text{ for } j \geq \mathcal{K}+1, u_j(t) = \int_\Omega u(x, t) X_j(x) dx\}$.

Here, we recall that $u(x, t)$ is said to be a weak solution of (1.1) with the initial value $u(x, 0) = \psi(x)$ if $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ and for any testing function $\varphi \in H_0^{1,1}(Q) = \{h \in L^2(Q); \partial_t h \in L^2(Q), D_i h \in L^2(Q) \text{ for all } i = 1, 2, \dots, n, h(x, t)|_{\partial\Omega} = 0\}$, we have, for all t ,

$$(u(\cdot, t), \varphi(\cdot, t)) - \int_0^t (u, \varphi_\tau) d\tau + \int_0^t (Lu, \varphi) d\tau = (\psi, \varphi(\cdot, 0)) + \int_0^t (f, \varphi) d\tau.$$

When $\mathcal{K} = 0$, we will always regard $\sum_{j=1}^0 = 0$. Hence, a 0-approximate periodic solution of (1.1) is a regular periodic solution.

It should be mentioned that in the above definition, we need only to assume that $u(x, t) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ to start with. Then it holds automatically that $u \in C([0, T]; L^2(\Omega))$ (see Chapter 3 of [5]). Also, we notice that

$$u \in \mathcal{S}_\mathcal{K} \iff (u - \sum_{j=1}^\mathcal{K} (u, X_j) X_j)(x, 0) = (u - \sum_{j=1}^\mathcal{K} (u, X_j) X_j)(x, T).$$

In the above formula and in what follows, we write $(u(\cdot, t), \varphi(\cdot, t)) = \int_\Omega u(x, t) \varphi(x, t) dx$, $(u(\cdot, t), u(\cdot, t)) = \|u(\cdot, t)\|^2$, and we denote u_t for the derivative of $u(x, t)$ with respect to t .

Our first result of this paper can be stated as follows:

Theorem 1.2. Let $e(x, t) \in \mathcal{M}_q = \{e(x, t) \in L^\infty(0, T; L^q(\Omega)); \text{ess sup}_{t \in (0, T)} \|e(x, t)\|_{L^q(\Omega)} \leq M, q > \frac{n+2}{2}, M \text{ is a constant}\}$. Then, there exists an integer $\mathcal{K}_0 \equiv \mathcal{K}_0(L_0, M, \Omega) \geq 0$ such that for any $\mathcal{K} \geq \mathcal{K}_0$ and any initial value $a_I = (a_1, a_2, \dots, a_\mathcal{K}) \in \mathbf{R}^\mathcal{K}$, we have a unique solution to the following equation:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + L_0 u(x, t) + e(x, t) u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma, \\ (u(x, 0), X_j(x)) = a_j, & \text{for } j \leq \mathcal{K}, \\ u \in \mathcal{S}_\mathcal{K}. \end{cases} \quad (1.4)$$

Moreover, for such a solution $u(x, t)$, we have the following energy estimate:

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|^2 + \int_0^T \|\nabla u(\cdot, t)\|^2 dt \leq C(L_0, M, \Omega)(|a_I|^2 + \int_Q f^2 dx dt). \quad (1.5)$$

The second part of this work is to study an inverse problem. We will identify $(e(x, t), a_I)$ from $\mathcal{M}_q \times \mathbf{R}^\mathcal{K}$ via the observation of solutions for (1.4) on the subdomain $\omega \subset \Omega$. More precisely, we shall study the following identification problem:

Problem (P) Find the minimum value of $\int_{Q^\omega} |u - \tilde{u}|^2 dxdt$ for $(e(x, t), a_I) \in \mathcal{M}_q \times \mathbf{R}^\mathcal{K}$ with u satisfying

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + L_0 u(x, t) + e(x, t)u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma, \\ (u(x, 0), X_j(x)) = a_j, & \text{for } j \leq \mathcal{K}, \\ u \in \mathcal{S}_\mathcal{K}, & \end{cases}$$

where $\tilde{u} \in L^2(Q^\omega)$ is a given function.

Making use of Theorem 1.2 and the Carleman inequality established in [2] [11] [16], etc, we are able to prove the existence of solutions to problem (P). Our second main result can be stated as follows:

Theorem 1.3. *Let \mathcal{K} be as in Theorem 1.2. Then there exist an $e^*(x, t) \in \mathcal{M}_q$ and $a_I^* \in \mathbf{R}^\mathcal{K}$ such that*

$$\int_{Q^\omega} |u(e^*, a_I^*; x, t) - \tilde{u}|^2 dxdt = \inf_{(e, a_I) \in \mathcal{M}_q \times \mathbf{R}^\mathcal{K}} \int_{Q^\omega} |u(e, a_I; x, t) - \tilde{u}|^2 dxdt.$$

Here $\tilde{u} \in L^2(Q^\omega)$ is a given function and $u(e, a_I; x, t)$ is the solution of equation (1.4) with error coefficient $e(x, t)$ and $(u(e, a_I; \cdot, 0), X_j(\cdot)) = a_j$ for $j \leq \mathcal{K}$, where $a_I = (a_1, \dots, a_\mathcal{K}) \in \mathbf{R}^\mathcal{K}$.

Theorem 1.3 can be immediately used to give the following slightly more general result:

Corollary 1.4. *Let k be a non-negative integer. Then there exist an $e^*(x, t) \in \mathcal{M}_q$ and $a_I^* \in \mathbf{R}^k$ such that*

$$\int_{Q^\omega} |u^* - \tilde{u}|^2 dxdt = \inf_{(e, a_I) \in \mathcal{M}_q \times \mathbf{R}^k, u \in U(e, a_I; x, t)} \int_{Q^\omega} |u - \tilde{u}|^2 dxdt.$$

Here $\tilde{u} \in L^2(Q^\omega)$ is a given function and $U(e, a_I; x, t)$ is the set of solutions of the following equation, which we assume to be non-empty:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + L_0 u(x, t) + e(x, t)u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma, \\ (u(x, 0), X_j(x)) = a_j, \quad a_I = (a_1, \dots, a_k), & \text{for } j \leq k, \\ u \in \mathcal{S}_k. & \end{cases} \quad (1.6)$$

Similarly, u^* satisfies the following equation:

$$\begin{cases} \frac{\partial u^*(x, t)}{\partial t} + L_0 u^*(x, t) + e^*(x, t) u^*(x, t) = f(x, t), & \text{in } Q, \\ u^*(x, t) = 0, & \text{on } \Sigma, \\ (u^*(x, 0), X_j(x)) = a_j^*, \quad a_I^* = (a_1^*, \dots, a_k^*), & \text{for } j \leq k, \\ u^* \in \mathcal{S}_k. \end{cases}$$

Notice that in Corollary 1.4, (1.6) may have a family of different solutions.

It is not clear to us if the uniqueness property for $e^*(x, t)$ in Theorem 1.3 holds. However, if one fixes $e(x, t)$ and tries to identify a_I through problem (P), then the uniqueness of a_I is indeed guaranteed as the following theorem shows:

Theorem 1.5. *Under the same notation as in Theorem 1.3, there exists a unique $a_I^* \in \mathbf{R}^\kappa$ such that*

$$\int_{Q^\omega} |u(e, a_I^*; x, t) - \tilde{u}|^2 dx dt = \inf_{a_I \in \mathbf{R}^\kappa} \int_{Q^\omega} |u(e, a_I; x, t) - \tilde{u}|^2 dx dt.$$

System (1.1) models a large class of physical processes, where $u(x, t)$ represents the temperature or other physical quantity. The identification problems associated with system (1.1) with initial condition $u(x, 0) = u_0(x)$, where $u_0(x)$ is a given function, were studied by many authors. See [1] [4] [6] [7] and [13], where the observations are taken in the whole domain Ω . However, in many applications, one may only be able to measure the quantity on a subdomain $\omega \subset \Omega$ and does not have enough information about the initial value. One may still be asked to determine the error influence $e(x, t)$ in the physical process through the approximate value of the solutions over ω . When the approximate value comes from approximate periodic solutions, then our results of the present paper can be directly applied. Notice that this is an inverse problem. For the direct problem, one is asked to determine the value of the solution to (1.1) for a given $e(x, t)$ and a_I .

Since our observation is taken in a subdomain $\omega \subset \Omega$, we can not apply the method employed in the work mentioned above to answer (P). Our key ingredients in this paper to get the existence of solutions for (P) are the energy estimate (1.5) and the Carleman inequality. There have been many papers written on the related subjects in recent years. Here we would like to mention [1-7] [14] [16] [17], and the reference therein, to name a few.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of the solution to system (1.4). In Section 3, we obtain the existence of the identification problem (P) by proving Theorem 1.3.

This work is a continuation of [14], where very special cases of the results in this paper were studied. It should be mentioned that the paper is largely motivated by two papers of G. Wang and L. Wang [16] [17].

2 The existence and uniqueness of the solution

In this section, we prove the existence and uniqueness of the solution to system (1.4). We will use the Galerkin method for constructing solutions in the \mathcal{S}_K -space for the following equation introduced in Section 1:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + L_0 u(x, t) + e(x, t)u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma. \end{cases} \quad (2.1)$$

Recall that $L_0 X_j(x) = \lambda_j X_j(x)$, $\lambda_j \rightarrow \infty$, $X_j(x) \in H_0^1(\Omega)$. Let $G_N = \{g(x, t) \in L^2(Q); g(x, t) = \sum_{j=1}^N g_j(t) X_j(x), g_j(t) \in L^2(0, T)\}$. We first look for an approximate solution $u^N(x, t)$ of (2.1) in the G_N -space, which also has the \mathcal{K} -approximate periodicity as defined before. Here \mathcal{K} depends only on the L_0 , M , Ω and will be determined later. N is always assumed to be sufficiently large ($N \gg \mathcal{K}$).

Write $L = L_0 + e$. Assume $u^N = \sum_{j=1}^N u_j^N(t) X_j(x)$ such that $\partial_t u^N = -Lu^N + f$ has 0 projection to G_N in the following sense:

$$(\partial_t u^N + Lu^N - f, \varphi) = 0, \quad \text{for } 0 < t < T \text{ and any } \varphi \in G_N.$$

Letting $\varphi = X_j$ for $j = 1, 2, \dots, N$, we get the following system of ordinary differential equations:

$$\begin{aligned} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) &= f_j(t), \quad j = 1, 2, \dots, N, \\ \text{where } B_{kj}(t) &= (LX_k, X_j) = \int_{\Omega} LX_k \cdot X_j dx, \quad f_j(t) = (f, X_j) = \int_{\Omega} f(x, t) X_j(x) dx. \end{aligned}$$

We put the following condition on u_j^N :

$$u_j^N(0) = u_j^N(T) \quad \text{for } j > \mathcal{K} \text{ with } \mathcal{K} \text{ independent of } N \text{ and being determined later.}$$

Consider the following system of ordinary differential equations:

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = 0, \quad j = 1, \dots, N, \\ u_I^N(0) = 0, \\ u_{II}^N(0) = a_{II}^N \in \mathbf{R}^{N-\mathcal{K}}. \end{cases} \quad (2.2)$$

Here and in what follows,

$$\begin{aligned} u_I^N(t) &= (u_1^N(t), u_2^N(t), \dots, u_{\mathcal{K}}^N(t)), \\ u_{II}^N(t) &= (u_{\mathcal{K}+1}^N(t), u_{\mathcal{K}+2}^N(t), \dots, u_N^N(t)). \end{aligned}$$

Lemma 2.1. *Let $u^N(x, t) = \sum_{j=1}^N u_j^N(t) X_j(x)$ be the solution of (2.2). There exists an integer \mathcal{K} depending only on L_0, M, Ω such that for any fixed $N > \mathcal{K}$, the operator:*

$$J : \mathbf{R}^{N-\mathcal{K}} \mapsto \mathbf{R}^{N-\mathcal{K}}, \quad J(a_{II}^N) = u_{II}^N(T),$$

is contractive. Namely,

$$|J(a_{II}^N)| \leq \mu |a_{II}^N| \quad \text{with } \mu \text{ fixed and } 0 \leq \mu < 1.$$

Here and in what follows, we always assume that $\text{ess sup}_{t \in (0, T)} \|e(x, t)\|_{L^q(\Omega)} \leq M$.

For the proof of Lemma 2.1, we need the following claim:

Claim 2.2. *For any $v_1, v_2 \in H_0^1(\Omega)$, there are constants $C(\Omega, q)$ depending only on Ω, q and $C_s(\varepsilon)$ and $C_l(\varepsilon)$, depending only on ε with $C_s(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, $C_l(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, such that*

$$\begin{aligned} \int_{\Omega} |e(x, t) v_1(x) v_2(x)| dx &\leq C(\Omega, q) M \{ C_l(\varepsilon) (\|v_1\|_{L^2(\Omega)}^2 + \|v_2\|_{L^2(\Omega)}^2) \\ &\quad + C_s(\varepsilon) (\|\nabla v_1\|_{L^2(\Omega)}^2 + \|\nabla v_2\|_{L^2(\Omega)}^2) \}. \end{aligned}$$

Proof of Claim 2.2. By the Schwartz inequality, we need only to prove the claim in the case of $v_1 = v_2 = v$.

$$\begin{aligned} \int_{\Omega} |e(x, t) v^2| dx &\leq \|e(\cdot, t)\|_{L^q(\Omega)} \|v^2\|_{L^{q'}(\Omega)} \\ &\leq M \|v\|_{L^{2q'}(\Omega)}^2, \end{aligned}$$

where $q' = \frac{q}{q-1}$. Let $\alpha = \frac{1}{2q'}(n+2-nq')$. Then $\frac{1}{2q'} = \frac{\alpha}{2} + \frac{1-\alpha}{\frac{2(n+2)}{n}}$. Next, by the Hölder inequality, we have

$$I = \int_{\Omega} |e(x, t) v^2| dx \leq M \|v\|_{L^2(\Omega)}^{2\alpha} \|v\|_{L^{\frac{2(n+2)}{n}}(\Omega)}^{2(1-\alpha)}.$$

By the Sobolev inequality,

$$\|v\|_{L^{\frac{2(n+2)}{n}}(\Omega)} \leq C(\Omega, q) \|v\|_{H_0^1(\Omega)}.$$

Hence, $I \leq MC(\Omega, q) (\|v\|_{L^2(\Omega)}^{\alpha} \|v\|_{H_0^1(\Omega)}^{1-\alpha})^2$.

Now, by the following Hölder inequality:

$$a \cdot b \leq \varepsilon a^p + \frac{1}{\varepsilon^q} b^q$$

with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} I &\leq MC(\Omega, q) (C_s(\varepsilon) \|v\|_{H_0^1(\Omega)} + C_l(\varepsilon) \|v\|_{L^2(\Omega)})^2 \\ &\leq MC(\Omega, q) (C_s(\varepsilon) \|v\|_{H_0^1(\Omega)}^2 + C_l(\varepsilon) \|v\|_{L^2(\Omega)}^2) \\ &\leq MC(\Omega, q) (C_s(\varepsilon) \|\nabla v\|_{L^2(\Omega)}^2 + C_l(\varepsilon) \|v\|_{L^2(\Omega)}^2). \end{aligned}$$

Here and in what follows, $C_s(\varepsilon), C_l(\varepsilon)$ stand for small and large constant depending only on ε , which may be different in different contexts. The proof of the claim is complete. ■

By Claim 2.2, we have

$$\begin{aligned} |B_{kj}(t)| &= |(L_0 X_k, X_j) + (e(x, t) X_k, X_j)| \\ &= |\lambda_j \delta_j^k + (e(x, t) X_k, X_j)| \\ &\leq |\lambda_j \delta_j^k| + |C_l(\varepsilon) + C_s(\varepsilon)(\|\nabla X_k\|_{L^2(\Omega)}^2 + \|\nabla X_j\|_{L^2(\Omega)}^2)|, \\ |f_j(t)| &\leq \left(\int_{\Omega} f^2(x, t) dx\right)^{\frac{1}{2}} \left(\int_{\Omega} X_j^2 dx\right)^{\frac{1}{2}} \leq \|f\|_{L^2(\Omega)}. \end{aligned}$$

In particular, we conclude that any solution of the initial value problem of

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = f_j(t), \quad j = 1, \dots, N, \\ (u_1^N(0), u_2^N(0), \dots, u_N^N(0)) = a^N \in \mathbf{R}^N, \end{cases} \quad (2.3)$$

is absolutely continuous over $[0, T]$.

Proof of Lemma 2.1. Multiplying $2u_j^N(t)$ to the first equation of (2.2) and summing up with respect to j from 1 to N , we get

$$\frac{d(\|u^N(\cdot, t)\|^2)}{dt} + 2(L_0 u^N, u^N) + 2(e(x, t) u^N, u^N) = 0.$$

As before, we use $\|\cdot\|$ to denote the usual $L^2(\Omega)$ -norm. After some calculation involving the Green formula, we have the following Gårding inequality (see [8]):

$$\frac{2}{\lambda^*} \|\nabla u^N\|^2 + C \|u^N\|^2 \geq (L_0 u^N, u^N) \geq \frac{\lambda^*}{2} \|\nabla u^N\|^2 - C \|u^N\|^2. \quad (2.4)$$

By (2.4) and Claim 2.2, we obtain

$$\frac{d(\|u^N(\cdot, t)\|^2)}{dt} + \lambda^* \|\nabla u^N\|^2 - C \|u^N\|^2 - C_s(\varepsilon) \|\nabla u^N\|^2 - C_l(\varepsilon) \|u^N\|^2 \leq 0.$$

We choose ε such that $C_s(\varepsilon) < \frac{\lambda^*}{2}$. Then we get, for a large constant C_l ,

$$\frac{d(\|u^N(\cdot, t)\|^2)}{dt} + \frac{\lambda^*}{2} \|\nabla u^N\|^2 - C_l(\varepsilon) \|u^N\|^2 \leq 0.$$

Applying the Gronwall inequality, we have

$$\begin{aligned} \frac{d}{dt} (\|u^N(\cdot, t)\|^2 e^{-C_l(\varepsilon)t}) + \frac{\lambda^*}{2} e^{-C_l(\varepsilon)t} \|\nabla u^N\|^2 &\leq 0, \\ \|u^N(\cdot, t)\|^2 + \int_0^t \|\nabla u^N(\cdot, \tau)\|^2 d\tau &\leq C \|u^N(\cdot, 0)\|^2, \quad \forall t \in [0, T]. \end{aligned}$$

In particular,

$$\|u^N(\cdot, T)\|^2 \leq C\|u^N(\cdot, 0)\|^2 = C|a_{II}^N|^2, \text{ and } \int_0^T \|\nabla u^N(\cdot, t)\|^2 dt \leq C|a_{II}^N|^2. \quad (2.5)$$

Next, multiplying $2u_j^N(t)$ to the first equation of (2.2) and summing up with respect to j from $\mathcal{K} + 1$ to N , and letting

$$u^{N,II}(x, t) = \sum_{j=\mathcal{K}+1}^N u_j^N(t) X_j(x),$$

then we get

$$\frac{d(\|u^{N,II}(\cdot, t)\|^2)}{dt} + 2(Lu^N, u^{N,II}) = 0.$$

Notice that

$$\begin{aligned} (Lu^N, u^{N,II}) &= (L_0 u^N, u^{N,II}) + (e(x, t) u^N, u^{N,II}) \\ &= (L_0 u^{N,II}, u^{N,II}) + (e(x, t) u^N, u^{N,II}) \\ &= \sum_{j=\mathcal{K}+1}^N \lambda_j (u_j^N(t))^2 + (e(x, t) u^N, u^{N,II}). \end{aligned}$$

By Claim 2.2 and (2.4), we have

$$\begin{aligned} |(e(x, t) u^N, u^{N,II})| &\leq C_l(\varepsilon)(\|u^N\|^2 + \|u^{N,II}\|^2) + C_s(\varepsilon)(\|\nabla u^N\|^2 + \|\nabla u^{N,II}\|^2) \\ &\leq C_l(\varepsilon)\|u^N\|^2 + C_s(\varepsilon)\|\nabla u^N\|^2 + C_s(\varepsilon)\left\{\frac{2C}{\lambda^*}\|u^{N,II}\|^2 + \frac{2}{\lambda^*}(L_0 u^{N,II}, u^{N,II})\right\} \\ &\leq C_l(\varepsilon)\|u^N\|^2 + C_s(\varepsilon)\|\nabla u^N\|^2 + C_s(\varepsilon)\left\{\frac{2C}{\lambda^*}\|u^N\|^2 + \frac{2}{\lambda^*} \sum_{j=\mathcal{K}+1}^N \lambda_j (u_j^N(t))^2\right\} \\ &\leq C_l(\varepsilon)\|u^N\|^2 + C_s(\varepsilon)\|\nabla u^N\|^2 + C_s(\varepsilon)\left\{\frac{2C}{\lambda^*}\|u^N\|^2 + \frac{2}{\lambda^*} \sum_{j=1}^N \lambda_j (u_j^N(t))^2\right\} \\ &\leq C_l(\varepsilon)\|u^N\|^2 + C_s(\varepsilon)\|\nabla u^N\|^2 + C_s(\varepsilon)\left\{\frac{2C}{\lambda^*}\|u^N\|^2 + \frac{2}{\lambda^*}(L_0 u^N, u^N)\right\} \\ &\leq C_l(\varepsilon)\|u^N\|^2 + C_s(\varepsilon)\|\nabla u^N\|^2 + C_s(\varepsilon)\left(\frac{4C}{\lambda^*}\|u^N\|^2 + \frac{4C}{\lambda^{*2}}\|\nabla u^N\|^2\right) \\ &\leq C_l(\varepsilon)\|u^N\|^2 + C_s(\varepsilon)\|\nabla u^N\|^2. \end{aligned}$$

Notice that to get the last inequality, we applied the other part of the Gårding estimate. We have

$$\frac{d(\|u^{N,II}(\cdot, t)\|^2)}{dt} + 2\lambda_{\mathcal{K}}\|u^{N,II}\|^2 - C_l(\varepsilon)\|u^N\|^2 - C_s(\varepsilon)\|\nabla u^N\|^2 \leq 0.$$

By the Gronwall inequality, we get

$$\begin{aligned} e^{2\lambda_{\mathcal{K}}t}\|u^{N,II}(\cdot, t)\|^2 - \|u^{N,II}(\cdot, 0)\|^2 &\leq C_l(\varepsilon) \int_0^t e^{2\lambda_{\mathcal{K}}\tau} \|u^N\|^2 d\tau \\ &\quad + C_s(\varepsilon) \int_0^t e^{2\lambda_{\mathcal{K}}\tau} \|\nabla u^N\|^2 d\tau, \quad \forall t \in [0, T]. \end{aligned}$$

We obtain

$$\begin{aligned} e^{2\lambda_{\mathcal{K}}T}\|u^{N,II}(\cdot, T)\|^2 - \|u^{N,II}(\cdot, 0)\|^2 &\leq C_l(\varepsilon) \int_0^T e^{2\lambda_{\mathcal{K}}t} \|u^N\|^2 dt \\ &\quad + C_s(\varepsilon) \int_0^T e^{2\lambda_{\mathcal{K}}t} \|\nabla u^N\|^2 dt, \end{aligned}$$

$$\begin{aligned}
\|u^{N,II}(\cdot, T)\|^2 &\leq e^{-2\lambda_K T} \|u^{N,II}(\cdot, 0)\|^2 + C_l(\varepsilon) \int_0^T e^{2\lambda_K(t-T)} \|u^N\|^2 dt \\
&\quad + C_s(\varepsilon) \int_0^T e^{2\lambda_K(t-T)} \|\nabla u^N\|^2 dt \\
&\leq e^{-2\lambda_K T} |a_{II}^N|^2 + C_l(\varepsilon) \cdot C \cdot |a_{II}^N|^2 \left(\frac{1}{2\lambda_K} - \frac{e^{-2\lambda_K T}}{2\lambda_K} \right) + C_s(\varepsilon) \cdot C \cdot |a_{II}^N|^2.
\end{aligned}$$

Now, we first choose ε sufficient small such that $C_s(\varepsilon) \cdot C < \frac{1}{4}$. Then we fix such an ε and fix a $K \gg 1$ such that $e^{-2\lambda_K T} < \frac{1}{4}$ and $C_l(\varepsilon) \cdot C \cdot \left(\frac{1}{2\lambda_K} - \frac{e^{-2\lambda_K T}}{2\lambda_K} \right) < \frac{1}{4}$. (Apparently, the choice of such a K depends only on the operator L_0, \bar{M}, Ω .) We then obtain

$$|u_{II}^N(T)|^2 = \|u^{N,II}(\cdot, T)\|^2 \leq \frac{3}{4} |a_{II}^N|^2. \quad (2.6)$$

Since $J(a_{II}^N) = u_{II}^N(T)$, we see the proof of Lemma 2.1. ■

Next, we prove the following proposition:

Proposition 2.3. *Let K be choose as above. Then for any $a_I^N \in \mathbf{R}^K$, there exists a unique solution u^N of the following mixed boundary value problem:*

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = f_j(t), & j = 1, \dots, N, \\ u_I^N(0) = a_I^N \in \mathbf{R}^K, \\ u_{II}^N(0) = u_{II}^N(T). \end{cases} \quad (2.7)$$

Proof of Proposition 2.3. Let $u_*^N(x, t) = \sum_{j=1}^N u_{*,j}^N(t) X_j(x)$ be the solution of the following system:

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = f_j(t), & j = 1, \dots, N, \\ (u_1^N(0), u_2^N(0), \dots, u_N^N(0)) = 0. \end{cases} \quad (2.8)$$

Let $u_{a^N}^N(x, t) = \sum_{j=1}^N u_{a^N,j}^N(t) X_j(x)$ be the solution of the following system:

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = 0, & j = 1, \dots, N, \\ (u_1^N(0), u_2^N(0), \dots, u_N^N(0)) = a^N = (a_I^N, a_{II}^N). \end{cases} \quad (2.9)$$

For a fixed $a_I^N \in \mathbf{R}^K$, we define

$$\tilde{J}_{a_I^N} : \mathbf{R}^{N-K} \longrightarrow \mathbf{R}^{N-K} \text{ such that } \tilde{J}_{a_I^N}(a_{II}^N) = u_{*II}^N(T) + u_{a^N II}^N(T),$$

where

$$\begin{aligned}
u_{*II}^N(T) &= (u_{*,K+1}^N(T), u_{*,K+2}^N(T), \dots, u_{*,N}^N(T)), \\
u_{a^N II}^N(T) &= (u_{a^N, K+1}^N(T), u_{a^N, K+2}^N(T), \dots, u_{a^N, N}^N(T)).
\end{aligned}$$

Namely, u_{*II}^N is the last $N - \mathcal{K}$ components of u_*^N and $u_{a^N II}^N$ is the last $N - \mathcal{K}$ components of $u_{a^N}^N$. We have

$$\begin{aligned} |\tilde{J}_{a_I^N}(a_{II}^{N,1}) - \tilde{J}_{a_I^N}(a_{II}^{N,2})| &= |u_{a^{N,1}II}^N(T) - u_{a^{N,2}II}^N(T)| \\ &= |J(a_{II}^{N,1} - a_{II}^{N,2})| \\ &\leq \frac{\sqrt{3}}{2} |a_{II}^{N,1} - a_{II}^{N,2}|. \end{aligned}$$

Hence, $\tilde{J}_{a_I^N}$ is a contractive map and has a unique fixed point $\widetilde{a_{II}^N}$. Namely, $\tilde{J}_{a_I^N}(\widetilde{a_{II}^N}) = \widetilde{a_{II}^N}$. Then (2.7) has a solution $u^N(x, t)$ with $u_{II}^N(0) = \widetilde{a_{II}^N} = u_{II}^N(T)$. The uniqueness also follows from the uniqueness of the fixed point of $\tilde{J}_{a_I^N}$. The proof of the proposition is complete. \blacksquare

Next, we estimate $\widetilde{a_{II}^N}$, the unique fixed point of $\tilde{J}_{a_I^N}$.

Proposition 2.4. *Let $\widetilde{a_{II}^N}$ be as above. Then $|\widetilde{a_{II}^N}|^2 \leq C(|a_I^N|^2 + \int_Q f^2 dx dt)$, where C depends only on L_0, M, Ω .*

Proof of Proposition 2.4. We know $\widetilde{a_{II}^N} = u_{II}^N(T) = u_{*,II}^N(T) + u_{a^N,II}^N(T)$. We first estimate $u_{*,II}^N(T)$. Multiplying the first equation of (2.8) by $2u_{*,j}^N(t)$ and summing up with respect to j from 1 to N , we have

$$\begin{aligned} \frac{d(\|u_*^N(\cdot, t)\|^2)}{dt} + 2(Lu_*^N, u_*^N) &= \sum_{j=1}^N f_j(t) u_{*,j}^N(t) \\ &\leq \|f\|^2 + \|u_*^N(\cdot, t)\|^2. \end{aligned}$$

By using the Gronwall inequality as before, we get

$$\sup_{t \in [0, T]} \|u_*^N(\cdot, t)\|^2 + \int_0^T \|\nabla u_*^N(\cdot, t)\|^2 dt \leq C \int_Q f^2 dx dt.$$

In particular, $\|u_*^N(\cdot, T)\|^2 \leq C \int_Q f^2 dx dt$. Hence,

$$|u_{*,II}^N(T)|^2 = \|u_{*,II}^N(\cdot, T)\|^2 \leq C \int_Q f^2 dx dt. \quad (2.10)$$

Next, we let $u^{N,1}(x, t), u^{N,2}(x, t)$ be the solution of the following system (2.11) and (2.12), respectively,

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = 0, & j = 1, \dots, N, \\ (u_1^N(0), u_2^N(0), \dots, u_N^N(0)) = (a_I^N, 0), \end{cases} \quad (2.11)$$

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t) u_k^N(t) = 0, & j = 1, \dots, N, \\ (u_1^N(0), u_2^N(0), \dots, u_N^N(0)) = (0, \widetilde{a_{II}^N}). \end{cases} \quad (2.12)$$

From the proof of Proposition 2.3 and Lemma 2.1, we get

$$\begin{aligned}
|u_{a^N II}^N(T)|^2 &= |u_{II}^{N,1}(T) + u_{II}^{N,2}(T)|^2 \\
&\leq C_l(\varepsilon)|a_I^N|^2 + (1 + C_s(\varepsilon))|J(\widetilde{a_{II}^N})|^2 \\
&\leq C_l|a_I^N|^2 + (1 + C_s)\frac{3}{4}|\widetilde{a_{II}^N}|^2.
\end{aligned} \tag{2.13}$$

By (2.10) and (2.13), we have

$$|\widetilde{a_{II}^N}|^2 \leq C(|a_I^N|^2 + \int_Q f^2 dx dt).$$

The proof of the proposition is complete. ■

Remark 2.5. From the proof of Lemma 2.1, we conclude that the value \mathcal{K} depends only on L_0 , Ω and M . Namely, given L_0 and M with $\text{ess sup}_{t \in (0, T)} \|e(x, t)\|_{L^q(\Omega)} \leq M$, there is a $\mathcal{K}_0 \equiv \mathcal{K}_0(L_0, M, \Omega)$ such that for any $N > \mathcal{K} \geq \mathcal{K}_0$ and $a_I^N = (a_1^N, a_2^N, \dots, a_K^N) \in \mathbf{R}^{\mathcal{K}}$, the following mixed value problem has a unique absolutely continuous solution $u^N(x, t) = \sum_{j=1}^N u_j^N(t) X_j(x)$ over $[0, T]$:

$$\begin{cases} \frac{du_j^N(t)}{dt} + \sum B_{kj}(t)u_k^N(t) = f_j(t), & j = 1, \dots, N, \\ u_I^N(0) = a_I^N, \\ u_{II}^N(0) = u_{II}^N(T). \end{cases} \tag{2.14}$$

Moreover, write $a_{II}^N = u_{II}^N(0) = u_{II}^N(T)$, we have the estimate:

$$|a_{II}^N|^2 \leq C(|a_I^N|^2 + \int_Q f^2 dx dt), \tag{2.14}'$$

where C depends only on L_0 , M , Ω .

Now, we follow the standard method to provide a convergence proof for the \mathcal{K} -approximate periodic solution $u^N(x, t)$ in Proposition 2.3. Since some minor changes are needed, we give some details. We first recall the energy estimate (Chapter 3 of [5]):

$$\sup_{t \in [0, T]} \|u^N(\cdot, t)\|^2 + \int_0^T \|\nabla u^N(\cdot, t)\|^2 dt \leq C(\|u^N(\cdot, 0)\|^2 + \int_Q f^2 dx dt).$$

By the estimate in (2.14)', we get

$$\sup_{t \in [0, T]} \|u^N(\cdot, t)\|^2 + \int_0^T \|\nabla u^N(\cdot, t)\|^2 dt \leq C(|a_I^N|^2 + \int_Q f^2 dx dt), \tag{2.15}$$

where C depends only on L_0, M, Ω , and where $u^N(x, t)$ is a solution of (2.7). We next fix $a_I^N = a_I = (a_1, a_2, \dots, a_K)$, and also \mathcal{K} that satisfies the property in Remark 2.5. We compute

$$\begin{aligned}
|u_j^N(t + \Delta t) - u_j^N(t)| &= \left| \int_t^{t+\Delta t} (\sum B_{kj}(\tau) u_k^N(\tau) - f_j(\tau)) d\tau \right| \\
&\leq \left| \int_t^{t+\Delta t} (Lu^N, X_j) d\tau \right| + \left| \int_t^{t+\Delta t} f_j(\tau) d\tau \right| \\
&\leq \left| \int_t^{t+\Delta t} \{C_l(\varepsilon) \|\nabla X_j\|^2 + C_s(\varepsilon) \|\nabla X_j\|^2 + C_s(\varepsilon) \|\nabla u^N\|^2 + C\} d\tau \right| \\
&\quad + \left(\int_t^{t+\Delta t} 1 \cdot d\tau \right)^{\frac{1}{2}} \left(\int_t^{t+\Delta t} f_j^2(\tau) d\tau \right)^{\frac{1}{2}} \\
&\leq C_s(\varepsilon) + C_l(\varepsilon) \sqrt{\Delta t}.
\end{aligned}$$

Now for any $\varepsilon' > 0$, we can find $\varepsilon > 0$ such that $C_s(\varepsilon) < \frac{\varepsilon'}{2}$. Then fixing such an ε , we can find $\delta > 0$ such that when $|\Delta t| < \delta$, $C_l(\varepsilon) \sqrt{\Delta t} < \frac{\varepsilon'}{2}$. Hence, when $|\Delta t| < \delta$,

$$|u_j^N(t + \Delta t) - u_j^N(t)| < \varepsilon'.$$

Note that δ is independent of N . We proved that $\{u_j^N(t)\}_{N=1}^\infty$ is equi-continuous. Since for any $N \gg 1$, $\sum_{j=1}^N |u_j^N(t)|^2 \leq \|u^N(\cdot, t)\|^2 \leq C$, $\{u_j^N(t)\}_{N=1}^\infty$ is uniformly bounded. Now by the Ascoli-Arzelà Theorem and the diagonal-element picking method, we can find a subsequence $\{N_l\}$ such that for each j , $u_j^{N_l}(t) \rightarrow u_j(t)$ uniformly over $[0, T]$.

Remark 2.6. We note that for each j , $\{u_j^N(t)\}_{N=1}^\infty$ is an equi-continuous family and uniformly bounded. For each j and for any $\varepsilon > 0$, there is a $\delta(j, \varepsilon)$ with δ depending only on j, L_0, M, ε such that when $|\Delta t| < \delta(j, \varepsilon)$, we have $|u_j(t + \Delta t) - u_j(t)| < \varepsilon$.

Next, for any fixed $m < N_l$, we have

$$\sum_{j=1}^m (u_j^{N_l}(t))^2 \leq \sum_{j=1}^{N_l} (u_j^{N_l}(t))^2 = \|u^{N_l}(\cdot, t)\|^2 \leq C.$$

Letting $N_l \rightarrow \infty$, we get $\sum_{j=1}^m (u_j(t))^2 \leq C$, and thus $\sum_{j=1}^\infty (u_j(t))^2 \leq C$. Let $u(x, t) = \sum_{j=1}^\infty u_j(t) X_j(x)$. We have $u(x, t) \in L^\infty(0, T; L^2(\Omega))$. Now, for $j \leq K$,

$$u_j(0) = \lim_{N_l \rightarrow \infty} u_j^{N_l}(0) = \lim_{N_l \rightarrow \infty} a_j = a_j.$$

For $j > K$,

$$u_j(0) = \lim_{N_l \rightarrow \infty} u_j^{N_l}(0) = \lim_{N_l \rightarrow \infty} u_j^{N_l}(T) = u_j(T).$$

Since $u^{N_l}(x, t), \nabla u^{N_l}(x, t) \in L^2(Q)$ with $\|u^{N_l}(x, t)\|_{L^2(Q)} \leq C$, $\|\nabla u^{N_l}(x, t)\|_{L^2(Q)} \leq C$, without loss of generality, we can assume that $u^{N_l}(x, t) \rightarrow u^*(x, t)$ weakly in $L^2(0, T; H_0^1(\Omega))$.

(See page 54 of [5]).

Let $\theta_j(t) \in C^\infty[0, T]$, and $\Phi^r(x, t) = \sum_{j=1}^r \theta_j(t) X_j(x)$, $N_l > r$. Since

$$\begin{aligned} \int_0^T (u^*(\cdot, t), \Phi^r(\cdot, t)) dt &= \lim_{N_l \rightarrow \infty} \int_0^T (u^{N_l}(\cdot, t), \Phi^r(\cdot, t)) dt \\ &= \lim_{N_l \rightarrow \infty} \int_0^T \sum_{j=1}^r u_j^{N_l}(t) \theta_j(t) dt = \int_0^T \sum_{j=1}^r u_j(t) \theta_j(t) dt \\ &= \int_0^T (u(\cdot, t), \Phi^r(\cdot, t)) dt. \end{aligned}$$

Since the set of all such $\Phi^r(x, t)$'s is dense in $L^2(0, T; H_0^1(\Omega))$, we get $u(x, t) \equiv u^*(x, t)$ a.e. in $L^2(0, T; H_0^1(\Omega))$. Hence, $u(x, t) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.

Now we prove $u(x, t)$ is a weak solution.

Let $\Phi^r(x, t) = \sum_{j=1}^r \theta_j(t) X_j(x)$ be as above and $N_l > r$. We have

$$\begin{aligned} \lim_{N_l \rightarrow \infty} (u^{N_l}(\cdot, t), \Phi^r(\cdot, t)) &= \lim_{N_l \rightarrow \infty} \sum_{j=1}^r u_j^{N_l}(t) \theta_j(t) = \sum_{j=1}^r u_j(t) \theta_j(t) = (u(\cdot, t), \Phi^r(\cdot, t)), \\ \lim_{N_l \rightarrow \infty} \int_0^t (u^{N_l}(\cdot, \tau), \Phi_\tau^r(\cdot, \tau)) d\tau &= \int_0^t \lim_{N_l \rightarrow \infty} \sum_{j=1}^r u_j^{N_l}(\tau) \theta_j'(\tau) d\tau \\ &= \int_0^t \sum_{j=1}^r u_j(\tau) \theta_j'(\tau) d\tau = \int_0^t (u(\cdot, \tau), \Phi_\tau^r(\cdot, \tau)) d\tau, \\ \lim_{N_l \rightarrow \infty} \int_0^t (Lu^{N_l}(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau &= \int_0^t (Lu(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau. \end{aligned}$$

Therefore, we get

$$\begin{aligned} (u(\cdot, t), \Phi^r(\cdot, t)) &- \int_0^t (u(\cdot, \tau), \Phi_\tau^r(\cdot, \tau)) d\tau + \int_0^t (Lu(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau \\ &= (u(\cdot, 0), \Phi^r(\cdot, 0)) + \int_0^t (f(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau. \end{aligned}$$

Since all such $\Phi^r(x, t)$'s are dense in $H_0^{1,1}(Q)$, we proved that $u(x, t)$ is a weak solution of (2.1). Note that $u(x, t) \in \mathcal{S}_K$. Summarizing the above, we proved the following:

Theorem 2.7. *There exists an integer $K_0 \equiv K_0(L_0, M, \Omega) \geq 0$ such that for any $K \geq K_0$ and any initial value $a_I = (a_1, a_2, \dots, a_K) \in \mathbf{R}^K$, there is a unique solution to the following equation:*

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + L_0 u(x, t) + e(x, t) u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma, \\ (u(\cdot, 0), X_j) = a_j, & j \leq K, \\ (u(\cdot, 0), X_j) = (u(\cdot, T), X_j), & j > K. \end{cases} \quad (2.16)$$

Theorem 2.8. *Let $u(x, t)$ be as in Theorem 2.7. Then there is a constant C depending only on L_0, M, Ω such that*

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|^2 + \int_0^T \|\nabla u(\cdot, t)\|^2 dt \leq C(|a_I|^2 + \int_Q f^2 dx dt). \quad (2.17)$$

Proof of Theorem 2.8. By (2.15), we have

$$\sup_{t \in [0, T]} \|u^{N_l}(\cdot, t)\|^2 + \int_0^T \|\nabla u^{N_l}(\cdot, t)\|^2 dt \leq C(|a_I|^2 + \int_Q f^2 dx dt).$$

Since $u^{N_l}(x, t) \rightarrow u^*(x, t)$ weakly in $L^2(0, T; H_0^1(\Omega))$, we have

$$\int_0^T \|\nabla u\|^2 dt \leq \lim_{N_l \rightarrow \infty} \int_0^T \|\nabla u^{N_l}\|^2 dt \leq C(|a_I|^2 + \int_Q f^2 dx dt). \quad (2.18)$$

$$\|u^{N_l}(\cdot, t)\|^2 = \sum_{j=1}^{N_l} (u_j^{N_l}(t))^2 \leq C(|a_I|^2 + \int_Q f^2 dx dt).$$

For any $m < N_l$, we have

$$\sum_{j=1}^m (u_j^{N_l}(t))^2 \leq C(|a_I|^2 + \int_Q f^2 dx dt).$$

Letting $N_l \rightarrow \infty$, we get $\sum_{j=1}^m (u_j(t))^2 \leq C(|a_I|^2 + \int_Q f^2 dx dt)$. Hence,

$$\sum_{j=1}^{\infty} (u_j(t))^2 = \|u(\cdot, t)\|^2 \leq C(|a_I|^2 + \int_Q f^2 dx dt). \quad (2.19)$$

By (2.18) and (2.19), we have (2.17). The proof is complete. ■

Proof of Theorem 2.7. It suffices to prove the uniqueness part of Theorem 2.7. Indeed, we need only to show that the only solution $u(x, t)$ of (2.16) with $f = 0, a_I = 0$ is 0. For this purpose, we first recall the energy estimate:

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|^2 + \int_0^T \|\nabla u(\cdot, t)\|^2 dt \leq C|a_{II}|^2, \quad (2.20)$$

where

$$a_{II} = ((u(\cdot, 0), X_{\mathcal{K}+1}), (u(\cdot, 0), X_{\mathcal{K}+2}), \dots).$$

Multiplying $X_j(x)$ to both sides of the first equation of (2.16) and then integrating over Ω , we get

$$\frac{du_j(t)}{dt} + (Lu, X_j) = 0. \quad (2.21)$$

Write $u^m(x, t) = \sum_{j=\mathcal{K}+1}^m (u, X_j) X_j(x) = \sum_{j=\mathcal{K}+1}^m u_j(t) X_j(x)$, $m > \mathcal{K}+1$. Then $u^m(x, 0) = u^m(x, T)$.

Multiplying $2u_j(t)$ to (2.21) and summing up with respect to j from $\mathcal{K}+1$ to m , we get

$$\frac{d(\|u^m(\cdot, t)\|^2)}{dt} + 2(L_0 u, u^m) + 2(e(x, t)u, u^m) = 0.$$

By the same arguments as those in the proof of Lemma 2.1, we have

$$\frac{d(\|u^m(\cdot, t)\|^2)}{dt} + 2\lambda_{\mathcal{K}}\|u^m(\cdot, t)\|^2 \leq C_s(\varepsilon)\|\nabla u\|^2 + C_l(\varepsilon)\|u\|^2.$$

Using the Gronwall inequality and noticing that $u^m(x, 0) = u^m(x, T)$, we get

$$\begin{aligned} (e^{2\lambda_{\mathcal{K}}T} - 1)\|u^m(\cdot, 0)\|^2 &\leq C_s(\varepsilon) \int_0^T e^{2\lambda_{\mathcal{K}}t} \|\nabla u\|^2 dt + C_l(\varepsilon) \int_0^T e^{2\lambda_{\mathcal{K}}t} \|u\|^2 dt \\ &\leq C_s(\varepsilon) e^{2\lambda_{\mathcal{K}}T} |a_{II}|^2 + C_l(\varepsilon) \frac{(e^{2\lambda_{\mathcal{K}}T} - 1)}{2\lambda_{\mathcal{K}}} |a_{II}|^2. \end{aligned}$$

So

$$\|u^m(\cdot, 0)\|^2 \leq C_s(\varepsilon) \frac{e^{2\lambda_{\mathcal{K}}T}}{e^{2\lambda_{\mathcal{K}}T} - 1} |a_{II}|^2 + C_l(\varepsilon) \frac{1}{2\lambda_{\mathcal{K}}} |a_{II}|^2.$$

Letting $m \rightarrow \infty$, we obtain

$$\lim_{m \rightarrow \infty} \|u^m(\cdot, 0)\|^2 = \lim_{m \rightarrow \infty} \sum_{j=\mathcal{K}+1}^m (u_j(0))^2 = |a_{II}|^2 \leq C_s(\varepsilon) |a_{II}|^2 + C_l(\varepsilon) \frac{1}{2\lambda_{\mathcal{K}}} |a_{II}|^2.$$

We first choose ε such that $C_s(\varepsilon) < \frac{1}{4}$, then we choose $\mathcal{K} \geq \mathcal{K}_0 \gg 1$ such that $C_l(\varepsilon) \frac{1}{2\lambda_{\mathcal{K}}} < \frac{1}{4}$. We have

$$|a_{II}|^2 \leq \frac{1}{4} |a_{II}|^2 + \frac{1}{4} |a_{II}|^2.$$

It says $|a_{II}|^2 \equiv 0$. By (2.20), we apparently get $u(x, t) \equiv 0$. The proof is complete. ■

Proof of Theorem 1.2. Theorem 1.2 follows directly from Theorem 2.7-2.8. ■

3 Existence of the solution to (P)

In this section, we give a proof of Theorem 1.3. Besides results established in §2, another main ingredient to be used here is the Carleman inequality for linear parabolic equation developed in [2] [10] and [16], which in particular implies the unique continuation property for the solutions.

First, by Theorem 2.7, there exists an integer $\mathcal{K}_0 \equiv \mathcal{K}_0(L_0, M, \Omega) \geq 0$ such that for any $\mathcal{K} \geq \mathcal{K}_0$ and any initial value $a_I = (a_1, a_2, \dots, a_{\mathcal{K}}) \in \mathbf{R}^{\mathcal{K}}$, we have a unique solution $u(x, t)$ to the following equation:

$$\begin{cases} \frac{\partial u(x, t)}{\partial t} + L_0 u(x, t) + e(x, t)u(x, t) = f(x, t), & \text{in } Q, \\ u(x, t) = 0, & \text{on } \Sigma, \\ (u(\cdot, 0), X_j) = a_j, & j \leq \mathcal{K}, \\ (u(\cdot, 0), X_j) = (u(\cdot, T), X_j), & j > \mathcal{K}. \end{cases} \quad (3.1)$$

For the proof of Theorem 1.3, we need the following Lemma:

Lemma 3.1. *Let $e_m \in \mathcal{M}_q$, $a_I^m = (a_1^m, a_2^m, \dots, a_{\mathcal{K}}^m) \in \mathbf{R}^{\mathcal{K}}$ with $m = 1, 2, \dots$. And let u_m ($m = 1, 2, \dots$) be the solution of the following:*

$$\begin{cases} \frac{\partial u_m(x, t)}{\partial t} + L_0 u_m(x, t) + e_m(x, t)u_m(x, t) = f_m(x, t), & \text{in } Q, \\ u_m(x, t) = 0, & \text{on } \Sigma, \\ (u_m(\cdot, 0), X_j) = a_j^m, & j \leq \mathcal{K}, \\ (u_m(\cdot, 0), X_j) = (u_m(\cdot, T), X_j), & j > \mathcal{K}. \end{cases} \quad (3.2)$$

Assume that $|a_I^m| \leq M_0$ with M_0 independent of the choice of m . Suppose that $e_m \rightarrow e^* \in \mathcal{M}_q$ in the weak-star topology of $L^\infty(0, T, L^q(\Omega))$, $a_j^m \rightarrow a_j^*$ for $j = 1, 2, \dots, \mathcal{K}$, and $f_m \rightarrow f^*$ in the $L^2(Q)$ -norm. Then there is a subsequence $\{m_k\}$ such that $\{u_{m_k}\}$ converges in the weak L^2 -topology to $u^* \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ with

$$\begin{cases} \frac{\partial u^*(x, t)}{\partial t} + L_0 u^*(x, t) + e^*(x, t)u^*(x, t) = f^*(x, t), & \text{in } Q, \\ u^*(x, t) = 0, & \text{on } \Sigma, \\ (u^*(\cdot, 0), X_j) = a_j^*, & j \leq \mathcal{K}, \\ (u^*(\cdot, 0), X_j) = (u^*(\cdot, T), X_j), & j > \mathcal{K}. \end{cases} \quad (3.3)$$

Moreover, write $u_{m_k}(x, t) = \sum_{j=1}^{\infty} u_j^{m_k}(t)X_j(x)$ and $u^*(x, t) = \sum_{j=1}^{\infty} u_j^*(t)X_j(x)$. Then by choosing m_k suitably, we have

$$\lim_{k \rightarrow \infty} u_j^{m_k}(t) = u_j^*(t), \text{ for any } t \in [0, T], j,$$

and for any $\delta > 0$, $u_{m_k}(x, t) \rightarrow u^*(x, t)$ strongly in the $L^2([\delta, T] \times \Omega)$ -norm.

Proof of Lemma 3.1. By the energy estimate in Theorem 2.7, we have

$$\sup_{t \in [0, T]} \|u_m(\cdot, t)\|^2 + \int_0^T \|\nabla u_m(\cdot, t)\|^2 dt \leq C(|a_I^m|^2 + \int_Q f_m^2 dx dt) \leq C,$$

where $a_I^m = (a_1^m, a_2^m, \dots, a_{\mathcal{K}}^m)$, C depends only on L_0, M , and Ω . By the assumption in Lemma 3.1, without loss of generality, we can assume $u_m(x, t) \rightarrow u^*(x, t)$, $\nabla u_m(x, t) \rightarrow$

$\nabla u^*(x, t)$ in the weak $L^2(Q)$ -topology. Apparently, $u^* \in L^2(Q) \cap L^2(0, T; H_0^1(\Omega))$.

By Remark 2.6, $\{u_j^m(t)\}_{m=1}^\infty$ is an equi-continuous family and uniformly bounded for each j . By the diagonal-element picking method and the Ascoli-Arzelà Theorem, we can find a subsequence $\{m_k\}$ such that

$$\text{for each } j, u_j^{m_k}(t) \rightarrow \widetilde{u}_j^*(t) \text{ uniformly over } [0, T].$$

Now, we let $\widetilde{u}^*(x, t) = \sum_{j=1}^\infty \widetilde{u}_j^*(t) X_j(x)$. Apparently, from the estimate:

$$\sum_{j=1}^N (u_j^{m_k}(t))^2 \leq C \text{ for any } N, \text{ it follows that } \sum_{j=1}^\infty (\widetilde{u}_j^*(t))^2 = \|\widetilde{u}^*(\cdot, t)\|^2 \leq C.$$

Now for any $\theta_j(t) \in C^\infty[0, T]$, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_0^T (u_{m_k}(\cdot, t), X_j) \theta_j(t) dt &= \lim_{k \rightarrow \infty} \int_0^T u_j^{m_k}(t) \theta_j(t) dt \\ &= \int_0^T \widetilde{u}_j^*(t) \theta_j(t) dt = \int_0^T (\widetilde{u}^*(\cdot, t), X_j) \theta_j(t) dt. \end{aligned}$$

On the other hand,

$$\lim_{k \rightarrow \infty} \int_0^T (u_{m_k}(\cdot, t), X_j) \theta_j(t) dt = \int_0^T (u^*(\cdot, t), X_j) \theta_j(t) dt.$$

Therefore,

$$\begin{aligned} \int_0^T (u^*(\cdot, t) - \widetilde{u}^*(\cdot, t), X_j \theta_j(t)) dt &= 0, \quad 0 = \int_0^T (u^*(\cdot, t) - \widetilde{u}^*(\cdot, t), \sum_{j=1}^r X_j) \theta_j(t) dt \\ &= \int_Q (u^*(x, t) - \widetilde{u}^*(x, t)) (\sum_{j=1}^r X_j(x) \theta_j(t)) dx dt \\ &= \int_Q (u^*(x, t) - \widetilde{u}^*(x, t)) \Phi^r(x, t) dx dt. \end{aligned}$$

Since the set of functions with the form $\Phi^r(x, t) = \sum_{j=1}^r \theta_j(t) X_j(x)$ is dense in $L^2(0, T; H_0^1(\Omega))$, we conclude that $\widetilde{u}^*(x, t) = u^*(x, t)$ a.e. in $L^2(0, T; H_0^1(\Omega))$. Therefore, $u^*(x, t) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$.

Next, for the $\Phi^r(x, t)$ defined above, by the assumption,

$$\begin{aligned} (u_m(\cdot, t), \Phi^r(\cdot, t)) &- \int_0^t (u_m(\cdot, \tau), \Phi_\tau^r(\cdot, \tau)) d\tau + \int_0^t (L_0 u_m(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau \\ &+ \int_0^t (e_m u_m(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau = (u_m(\cdot, 0), \Phi^r(\cdot, 0)) + \int_0^t (f_m(\cdot, \tau), \Phi^r(\cdot, \tau)) d\tau. \end{aligned}$$

Now, as in Section 2, to show that $u^*(x, t)$ is the weak solution of (3.3), it suffices to show that

$$\lim_{k \rightarrow \infty} \int_0^T \int_\Omega e_{m_k}(x, \tau) u_{m_k}(x, \tau) \Phi^r(x, \tau) dx d\tau = \int_0^T \int_\Omega e^*(x, \tau) u^*(x, \tau) \Phi^r(x, \tau) dx d\tau \quad (3.4)$$

for a certain subsequence $\{m_k\}$. Notice that for $j \leq \mathcal{K}$,

$$(u^*(\cdot, 0), X_j) = u_j^*(0) = \lim_{k \rightarrow \infty} u_j^{m_k}(0) = \lim_{k \rightarrow \infty} a_j^{m_k} = a_j^*;$$

for $j > \mathcal{K}$,

$$(u^*(\cdot, 0), X_j) = \lim_{k \rightarrow \infty} (u_{m_k}(\cdot, 0), X_j) = \lim_{k \rightarrow \infty} (u_{m_k}(\cdot, T), X_j) = (u^*(\cdot, T), X_j).$$

Hence, the proof of Lemma 3.1 will be complete if we can prove (3.4).

Next, notice that

$$\begin{aligned} \left| \int_Q e_{m_k} u_{m_k} \Phi^r dx dt - \int_Q e^* u^* \Phi^r dx dt \right| &\leq \left| \int_Q (e_{m_k} - e^*) u^* \Phi^r dx dt \right| \\ &\quad + \left| \int_Q e_{m_k} (u_{m_k} - u^*) \Phi^r dx dt \right|. \end{aligned}$$

Apparently, $\left| \int_Q (e_{m_k} - e^*) u^* \Phi^r dx dt \right| \rightarrow 0$. Therefore, it suffices to prove the following claim to complete the proof of Lemma 3.1.

Claim 3.2. There is a subsequence $\{m_k\}$ such that $\left| \int_Q e_{m_k} (u_{m_k} - u^*) \Phi^r dx dt \right| \rightarrow 0$, and for any $\delta > 0$,

$$u_{m_k}(x, t) \rightarrow u^*(x, t) \text{ strongly in the } L^2([\delta, T] \times \Omega), \text{ as } k \rightarrow \infty.$$

Proof of Claim 3.2. Notice that

$$\begin{cases} \frac{\partial(t^{\frac{2}{3}} u_m(x, t))}{\partial t} + L(t^{\frac{2}{3}} u_m(x, t)) = t^{\frac{2}{3}} f_m(x, t) + \frac{2}{3} t^{-\frac{1}{3}} u_m(x, t), & \text{in } Q = \Omega \times (0, T), \\ t^{\frac{2}{3}} u_m(x, t) = 0, & \text{on } \Sigma = \partial\Omega \times (0, T), \\ t^{\frac{2}{3}} u_m(x, t)|_{t=0} = 0, & \text{in } \Omega. \end{cases}$$

By the high order energy estimates for parabolic equations (page 59, Theorem 4.1 of [5]), we have

$$\begin{aligned} \sup_{t \in (0, T)} \|\nabla(t^{\frac{2}{3}} u_m(\cdot, t))\|^2 + \int_Q |\partial_t(t^{\frac{2}{3}} u_m(x, t))|^2 dx dt &\leq \int_Q |t^{\frac{2}{3}} f_m(x, t) + \frac{2}{3} t^{-\frac{1}{3}} u_m(x, t)|^2 dx dt \\ &\leq C. \end{aligned}$$

Then $\|t^{\frac{2}{3}} u_m(x, t)\|_{W^{1,1}(Q)} \leq C$ for all m . By the Rellich lemma, there is a subsequence $\{m_k\}$ such that $t^{\frac{2}{3}} u_{m_k}(x, t) \rightarrow \bar{u}(x, t)$ strongly in $L^2(Q)$. Next,

$$\begin{aligned} \int_Q \bar{u}(x, t) \Phi^r(x, t) dx dt &= \lim_{k \rightarrow \infty} \int_Q t^{\frac{2}{3}} u_{m_k}(x, t) \Phi^r(x, t) dx dt \\ &= \lim_{k \rightarrow \infty} \int_Q u_{m_k}(x, t) t^{\frac{2}{3}} \Phi^r(x, t) dx dt = \int_Q t^{\frac{2}{3}} u^*(x, t) \Phi^r(x, t) dx dt. \end{aligned}$$

Since all such $\Phi^r(x, t)'s$ are dense in $L^2(0, T; H_0^1(\Omega))$, $\bar{u}(x, t) = t^{\frac{2}{3}}u^*(x, t)$ a.e. in $L^2(0, T; H_0^1(\Omega))$. Now,

$$\begin{aligned} \int_{\delta}^T \int_{\Omega} |u_{m_k} - u^*|^2 dx dt &= \int_{\delta}^T \int_{\Omega} t^{-4/3} |t^{\frac{2}{3}}(u_{m_k} - u^*)|^2 dx dt \\ &\leq C(\delta) \int_{\delta}^T \int_{\Omega} |t^{\frac{2}{3}}(u_{m_k} - u^*)|^2 dx dt \leq C(\delta) \int_0^T \int_{\Omega} |t^{\frac{2}{3}}(u_{m_k} - u^*)|^2 dx dt \rightarrow 0. \end{aligned}$$

We thus see, for any $\delta > 0$, that

$$u_{m_k}(x, t) \rightarrow u^*(x, t) \text{ strongly in the } L^2([\delta, T] \times \Omega).$$

Next, by Claim 2.2,

$$\begin{aligned} \left| \int_{\Omega} e_{m_k}(u_{m_k} - u^*) \Phi^r dx \right| &\leq C_s(\varepsilon) \int_{\Omega} |e_{m_k}(u_{m_k} - u^*)|^2 dx + C_l(\varepsilon) \int_{\Omega} |e_{m_k}(\Phi^r)|^2 dx \\ &\leq C_s(\varepsilon) \left(\int_{\Omega} |\nabla(u_{m_k} - u^*)|^2 dx + \int_{\Omega} |u_{m_k} - u^*|^2 dx \right) + C_l(\varepsilon) \\ &\leq C_s(\varepsilon) \int_{\Omega} |\nabla(u_{m_k} - u^*)|^2 dx + C_l(\varepsilon). \end{aligned}$$

Next,

$$\begin{aligned} \left| \int_0^{\delta} \int_{\Omega} e_{m_k}(u_{m_k} - u^*) \Phi^r dx dt \right| &\leq \int_0^{\delta} (C_s(\varepsilon) \int_{\Omega} |\nabla(u_{m_k} - u^*)|^2 dx + C_l(\varepsilon)) dt \\ &\leq C_s(\varepsilon) + \delta C_l(\varepsilon). \end{aligned}$$

Now, for any $\varepsilon' > 0$, $\exists \varepsilon(\varepsilon')$ such that $C_s(\varepsilon) < \frac{\varepsilon'}{4}$. Then there is a $\delta = \delta(\varepsilon, \varepsilon')$ such that $\delta C_l(\varepsilon) < \frac{\varepsilon'}{4}$. We thus have

$$\left| \int_0^{\delta} \int_{\Omega} e_{m_k}(u_{m_k} - u^*) \Phi^r dx dt \right| \leq C_s(\varepsilon) + \delta C_l(\varepsilon) \leq \frac{\varepsilon'}{2}. \quad (3.5)$$

Notice that

$$\begin{aligned} \left| \int_{\delta}^T \int_{\Omega} e_{m_k}(u_{m_k} - u^*) \Phi^r dx dt \right| &\leq C_l(\varepsilon'') \int_{\delta}^T \int_{\Omega} |e_{m_k}(u_{m_k} - u^*)|^2 dx dt \\ &\quad + C_s(\varepsilon'') \int_{\delta}^T \int_{\Omega} |e_{m_k}(\Phi^r)|^2 dx dt \\ &\leq C_l(\varepsilon'') \{ C_l(\varepsilon''') \int_{\delta}^T \int_{\Omega} |u_{m_k} - u^*|^2 dx dt \\ &\quad + C_s(\varepsilon''') \int_{\delta}^T \int_{\Omega} |\nabla(u_{m_k} - u^*)|^2 dx dt \} + C_s(\varepsilon''). \end{aligned}$$

We have

$$\overline{\lim}_{k \rightarrow \infty} \left| \int_{\delta}^T \int_{\Omega} e_{m_k}(u_{m_k} - u^*) \Phi^r dx dt \right| \leq C_l(\varepsilon'') C_s(\varepsilon''') + C_s(\varepsilon'').$$

For the ε' as before, we can choose ε'' such that $C_s(\varepsilon'') < \frac{\varepsilon'}{4}$. Then for this fixed ε'' , there exists an ε''' such that $C_l(\varepsilon'')C_s(\varepsilon''') < \frac{\varepsilon'}{4}$. Hence

$$\overline{\lim}_{k \rightarrow \infty} \left| \int_{\delta}^T \int_{\Omega} e_{m_k}(u_{m_k} - u^*) \Phi^r dx dt \right| \leq \frac{\varepsilon'}{2}. \quad (3.6)$$

Since ε' is arbitrary, by (3.5) and (3.6), we get $\left| \int_Q e_{m_k}(u_{m_k} - u^*) \Phi^r dx dt \right| \rightarrow 0$, as $k \rightarrow \infty$. The proof of Lemma 3.1 is complete. ■

Proof of Theorem 1.3. Let $d = \inf_{(e, a_I) \in \mathcal{M}_q \times \mathbf{R}^{\mathcal{K}}} \int_{Q^\omega} |u(e, a_I; x, t) - \tilde{u}|^2 dx dt$. It is obvious that $d < \infty$. Thus there exists a sequence $\{(e_m, a_I^m)\}_{m=1}^\infty$ such that

$$d \leq \int_{Q^\omega} |u_m(e_m, a_I^m; x, t) - \tilde{u}|^2 dx dt \leq d + \frac{1}{m}, \quad (3.7)$$

and

$$\begin{cases} \frac{\partial u_m(e_m, a_I^m; x, t)}{\partial t} + L_0 u_m(e_m, a_I^m; x, t) + e_m(x, t) u_m(e_m, a_I^m; x, t) = f(x, t), & \text{in } Q, \\ u_m(e_m, a_I^m; x, t) = 0, & \text{on } \Sigma, \\ (u_m(e_m, a_I^m; \cdot, 0), X_j) = a_j^m, & j \leq \mathcal{K}, \\ (u_m(e_m, a_I^m; \cdot, 0), X_j) = (u_m(e_m, a_I^m; \cdot, T), X_j), & j > \mathcal{K}. \end{cases} \quad (3.8)$$

In what follows, when there is no confusion of notation, we simply write $u_m(x, t)$ for $u_m(e_m, a_I^m; x, t)$. By the definition of \mathcal{M}_q , there exists a subsequence $\{m_k\}$ and $e^* \in \mathcal{M}_q$ such that

$$e_{m_k} \rightarrow e^* \text{ in the weak star topology as } k \rightarrow \infty. \quad (3.9)$$

By (3.7),

$$\int_{Q^\omega} |u_m|^2 dx dt \leq \int_{Q^\omega} |\tilde{u}|^2 dx dt + C \leq C \quad (3.10)$$

with C independent of m .

Next, we prove the following claim:

Claim 3.3. *There is a constant M_0 such that $|a_I^m|^2 \leq M_0$ for all m .*

Proof of Claim 3.3. Suppose not. There is a subsequence $\{m_k\}$ such that $\mu_k = |a_I^{m_k}| \rightarrow \infty$ as $k \rightarrow \infty$. We write $\hat{u}_{m_k}(x, t) = \frac{u_{m_k}(x, t)}{\mu_k}$. Then

$$\begin{cases} \frac{\partial \hat{u}_{m_k}(x, t)}{\partial t} + L_0 \hat{u}_{m_k}(x, t) + e_{m_k}(x, t) \hat{u}_{m_k}(x, t) = \frac{f(x, t)}{\mu_k}, & \text{in } Q, \\ \hat{u}_{m_k}(x, t) = 0, & \text{on } \Sigma, \\ (\hat{u}_{m_k}(\cdot, 0), X_j) = \frac{a_j^{m_k}}{\mu_k}, & j \leq \mathcal{K}, \\ (\hat{u}_{m_k}(\cdot, 0), X_j) = (\hat{u}_{m_k}(\cdot, T), X_j), & j > \mathcal{K}. \end{cases} \quad (3.11)$$

Apparently, $|\frac{a_I^{m_k}}{\mu_k}|^2 = 1$ for each m_k . After passing to a subsequence, if necessary, we can assume that

$$\frac{a_j^{m_k}}{\mu_k} \rightarrow \hat{a}_j \text{ with } |\hat{a}_I| = 1, \hat{a}_I = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_K), j = 1, 2, \dots, K. \quad (3.12)$$

Then by Lemma 3.1, there is a $\hat{u} \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ such that $\hat{u}_{m_k} \rightarrow \hat{u}$ in the weak L^2 -topology, and for any $\delta > 0$, $\hat{u}_{m_k}(x, t) \rightarrow \hat{u}(x, t)$ strongly in the $L^2([\delta, T] \times \Omega)$ topology. Also, all the other statements in Lemma 3.1 hold.

By (3.7), we have

$$\begin{aligned} C &\geq \int_{Q^\omega} |u_{m_k}(x, t) - \tilde{u}|^2 dx dt \\ &= \int_{Q^\omega} |\mu_k \hat{u}_{m_k}(x, t) - \tilde{u}|^2 dx dt \\ &= (\mu_k)^2 \int_{Q^\omega} |\hat{u}_{m_k}(x, t) - \frac{\tilde{u}}{\mu_k}|^2 dx dt \\ &\geq (\mu_k)^2 \int_{[\delta, T] \times \omega} |\hat{u}_{m_k}(x, t) - \frac{\tilde{u}}{\mu_k}|^2 dx dt, \quad \forall \delta > 0. \end{aligned}$$

Since $\mu_k \rightarrow \infty$, we see that

$$\lim_{k \rightarrow \infty} \int_{[\delta, T] \times \omega} |\hat{u}_{m_k}(x, t) - \frac{\tilde{u}}{\mu_k}|^2 dx dt = 0, \quad \forall \delta > 0.$$

From the property that $\hat{u}_{m_k}(x, t) \rightarrow \hat{u}(x, t)$ strongly in the $L^2([\delta, T] \times \Omega)$, it follows that

$$\int_{[\delta, T] \times \omega} |\hat{u}(x, t)|^2 dx dt = 0, \quad \forall \delta > 0.$$

By the unique continuation property for solutions of the parabolic equations, which is a consequence the Carleman inequality (see Page 430, Inequality (2) of [16]), we get $\hat{u}(x, t) \equiv 0$ in $(0, T] \times \Omega$. Since $\hat{u} \in C([0, T], L^2(\Omega))$, we get $\hat{u}(x, 0) \equiv 0$. On the other hand, by (3.12), $(\hat{u}(\cdot, 0), X_j) = \hat{a}_I$ with $|\hat{a}_I| = 1$. We see a contraction. The proof of Claim 3.3 is complete. ■

Now, making use of Claim 3.3, Lemma 3.1 and (3.9), we can assume that there is a subsequence $\{m_k\}$ such that $a_I^{m_k} \rightarrow a_I^*$, $\{u_{m_k}\}$ converges in the weak L^2 -topology to $u^* \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ with

$$\begin{cases} \frac{\partial u^*(x, t)}{\partial t} + L_0 u^*(x, t) + e^*(x, t) u^*(x, t) = f^*(x, t), & \text{in } Q, \\ u^*(x, t) = 0, & \text{on } \Sigma, \\ (u^*(\cdot, 0), X_j) = a_j^*, & j \leq K, \\ (u^*(\cdot, 0), X_j) = (u^*(\cdot, T), X_j), & j > K. \end{cases}$$

By (3.7), we obtain

$$d \leq \int_{Q^\omega} |u^*(x, t) - \tilde{u}|^2 dx dt \leq \lim_{k \rightarrow \infty} \int_{Q^\omega} |u_{m_k}(x, t) - \tilde{u}|^2 dx dt \leq d.$$

So

$$\int_{Q^\omega} |u(e^*, a_I^*; x, t) - \tilde{u}|^2 dx dt = \inf_{(e, a_I) \in \mathcal{M}_q \times \mathbf{R}^\mathcal{K}} \int_{Q^\omega} |u(e, a_I; x, t) - \tilde{u}|^2 dx dt.$$

The proof of Theorem 1.3 is complete. ■

Proof of Corollary 1.4: By Theorem 1.3, we can assume that $k < \mathcal{K}_0$, where \mathcal{K}_0 is the same integer as in Theorem 1.2. Notice that $\mathcal{S}_k \subset \mathcal{S}_{\mathcal{K}_0}$. We keep the notation set up before.

Still let $d = \inf_{(e, a_I) \in \mathcal{M}_q \times \mathbf{R}^k, u \in U(e, a_I; x, t)} \int_{Q^\omega} |u(x, t) - \tilde{u}|^2 dx dt$ with $d < \infty$. We then also have a sequence of the pairs $\{(e_m, a_I^m)\}_{m=1}^\infty$ and a sequence of functions $u_m(e_m, a_I^m; x, t)$ such that

$$d \leq \int_{Q^\omega} |u_m(e_m, a_I^m; x, t) - \tilde{u}|^2 dx dt \leq d + \frac{1}{m}, \quad (3.13)$$

and

$$\begin{cases} \frac{\partial u_m(e_m, a_I^m; x, t)}{\partial t} + L_0 u_m(e_m, a_I^m; x, t) + e_m(x, t) u_m(e_m, a_I^m; x, t) = f(x, t), & \text{in } Q, \\ u_m(e_m, a_I^m; x, t) = 0, & \text{on } \Sigma, \\ (u_m(e_m, a_I^m; \cdot, 0), X_j) = a_j^m, & j \leq k, \\ (u_m(e_m, a_I^m; \cdot, 0), X_j) = (u_m(e_m, a_I^m; \cdot, T), X_j), & j > k. \end{cases} \quad (3.14)$$

Now, write $\tilde{a}_I^m = (a_1^m, \dots, a_k^m, \tilde{a}_{k+1}^m, \tilde{a}_{k+2}^m, \dots, \tilde{a}_{\mathcal{K}_0}^m)$ with $\tilde{a}_j^m = (u_m(e_m, a_I^m; x, 0), X_j) = (u_m(e_m, a_I^m; x, T), X_j)$ for $j = k+1, k+2, \dots, \mathcal{K}_0$. Notice that

$$\int_{Q^\omega} |u_m(e_m, a_I^m; x, t)|^2 dx dt \leq \int_{Q^\omega} |\tilde{u}|^2 dx dt + C \leq C. \quad (3.15)$$

Since $u_m(e_m, a_I^m; x, t) \in \mathcal{S}_k \subset \mathcal{S}_{\mathcal{K}_0}$ for each m , making use of Lemma 3.1, we can repeat the same argument as in the proof of Theorem 1.3 to show that there is a subsequence $\{u_{m_l}(e_{m_l}, a_I^{m_l}; x, t)\}$ of $\{u_m(e_m, a_I^m; x, t)\}$ such that $\tilde{a}_I^{m_l} \rightarrow a_I^*$ and $\{u_{m_l}\}$ converges in the weak L^2 -topology to $u^* \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ with

$$\begin{cases} \frac{\partial u^*(x, t)}{\partial t} + L_0 u^*(x, t) + e^*(x, t) u^*(x, t) = f(x, t), & \text{in } Q, \\ u^*(x, t) = 0, & \text{on } \Sigma, \\ (u^*(\cdot, 0), X_j) = a_j^*, & j \leq \mathcal{K}_0, \\ (u^*(\cdot, 0), X_j) = (u^*(\cdot, T), X_j), & j > \mathcal{K}_0. \end{cases}$$

Moreover, as in Lemma 3.1, we also have

$$u_j^*(0) = (u^*(\cdot, 0), X_j) = \lim_{m_i \rightarrow \infty} u_j^{m_i}(0) = \lim_{m_i \rightarrow \infty} u_j^{m_i}(T) = u_j^*(T) \text{ for } j > k.$$

Hence, $u^* \in \mathcal{S}_k$. Now, by the same argument as in the proof of Theorem 1.3, we conclude the proof of Corollary 1.4. ■

Proof of Theorem 1.5: Keep the notation as in Theorem 1.5. Let $a_I^1, a_I^2 \in \mathbf{R}^\mathcal{K}$ be such that

$$d = \int_{Q^\omega} |u(e, a_I^j; x, t) - \tilde{u}|^2 dx dt = \inf_{a_I \in \mathbf{R}^\mathcal{K}} \int_{Q^\omega} |u(e, a_I; x, t) - \tilde{u}|^2 dx dt, \quad j = 1, 2.$$

For $\tau \in \mathbf{R}$, define

$$I(\tau) = \int_{Q^\omega} |u(e, \tau a_I^1 + (1 - \tau)a_I^2; x, t) - \tilde{u}|^2 dx dt. \quad \text{Then}$$

$$I(\tau) = \int_{Q^\omega} |\tau(u(e, a_I^1; x, t) - \tilde{u}) + (1 - \tau)(u(e, a_I^2; x, t) - \tilde{u})|^2 dx dt.$$

$$= d(\tau^2 + (1 - \tau)^2) + 2\tau(1 - \tau) \int_{Q^\omega} (u(e, a_I^1; x, t) - \tilde{u})(u(e, a_I^2; x, t) - \tilde{u}) dx dt.$$

Since $I(\tau)$ achieves its minimum value at $\tau = 0$, we have $I'(0) = 0$, from which the following follows:

$$d = \int_{Q^\omega} (u(e, a_I^1; x, t) - \tilde{u})(u(e, a_I^2; x, t) - \tilde{u}) dx dt.$$

On the other hand, by the Hölder inequality, we have

$$\begin{aligned} & \left(\int_{Q^\omega} (u(e, a_I^1; x, t) - \tilde{u})(u(e, a_I^2; x, t) - \tilde{u}) dx dt \right)^2 \\ & \leq \int_{Q^\omega} |u(e, a_I^1; x, t) - \tilde{u}|^2 dx dt \cdot \int_{Q^\omega} |u(e, a_I^2; x, t) - \tilde{u}|^2 dx dt = d^2, \end{aligned}$$

with equality being held if and only if $(u(e, a_I^1; x, t) - \tilde{u}) = C(u(e, a_I^2; x, t) - \tilde{u})$ over Q^ω for a certain constant C . Apparently, this implies that

$$u(e, a_I^1; x, t) - \tilde{u} = u(e, a_I^2; x, t) - \tilde{u} \quad \text{over } Q^\omega.$$

By the Carleman inequality as mentioned in the proof of Theorem 1.3, we conclude that $u(e, a_I^1; x, t) = u(e, a_I^2; x, t)$ over Q . This then forces that they have the same initial value. In particular we conclude that $a_I^1 = a_I^2$. The proof of Theorem 1.5 is complete. ■

Example 3.4: Consider the following heat equation:

$$y_t = \Delta y + cy + f(x, t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1. \quad (3.16)$$

Here $c \in \mathbf{R}$ and $f \in L^2((0, 1) \times (0, 1))$. Notice that $\{\frac{1}{\sqrt{2}} \sin(k\pi x)\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2(0, 1)$. Write $f(x, t) = \sum_{k=1}^{\infty} f_k(t) \sin(k\pi x)$ and $y = \sum_{k=1}^{\infty} a_k(t) \sin(k\pi x)$. Then we have $a'_k(t) = -(k\pi)^2 a_k(t) + c a_k(t) + f_k(t)$. Thus, we get

$$a_k(1)e^{-c+(k\pi)^2} - a_k(0) = \int_0^1 f_k(t)e^{(-c+(k\pi)^2)t} dt.$$

Now, choose $c = (K\pi)^2$ and choose f such that $\int_0^1 f_K(t) dt \neq 0$. Then (3.16) can never have a solution y , which is in the space \mathcal{S}_{K-1} . However, in this case, for any $b_I^K = (b_1, \dots, b_K)$, (3.16) does have a unique solution $y(x, t) \in \mathcal{S}_K$ with $(y(\cdot, 0), \frac{1}{\sqrt{2}} \sin(j\pi x)) = b_j$ for $j = 1, 2, \dots, K$.

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